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LIPSCHITZ LABELING OF SOME TREES

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Abstract

A Lipschitz labeling of a graph G with vertex set V(G) with $|V(G)| = m$ and edge set E(G) with $|E(G)| = n \geq 1$ is a function $f: V(G) \cup E(G) \to \{1, 2, \ldots, m+n\}$ which satisfy the following conditions.

(i) f is bijective.

(ii) $f(v) \in \{1, 2, \ldots, m\}$ for all $v \in V(G)$

(iii) $f(x) \in \{m+1, m+2, \ldots, m+n\}$ for all $x \in E(G)$

(iv) There exists a positive integer L such that

$$|f(x) - f(y)| \leq L|x_f - y_f|$$

for all $x, y \in E(G)$ and $x_f = \min\{f(u), f(v)\}$ for $x = uv$.

A graph which admits a Lipschitz labeling is called a Lipschitz graph. In this work, we focus on examining the existence of Lipschitz labeling for a fork graph with two prongs, an h graph, a perfect binary tree and a flag graph.

Keywords: *Finite Graph, Lipschitz Labeling, A Fork Graph, Binary Tree, A Flag Graph.*

1. INTRODUCTION

In graph theory, a graph labeling is the assignment of integer labels to the vertices, edges, or both within a graph, according to specified conditions. The now-standard term graceful labeling in graph theory describes a spe- cific and aesthetically pleasing assignment of integers to a graph's vertices. This involves a function f that labels the vertices of a graph G with distinct integers from $\{0, 1, 2, \ldots, q\}$ (where q is the edge count), such that the abso- lute differences $|f(x) - f(y)|$ for each edge xy are all distinct. Interestingly, this concept has two namesakes. It was first conceived by Rosa [7], who called it a β-valuation. Later, and independently, Golomb[4] studied the same structure, naming it a graceful labeling, which subsequently became the widely adopted terminology. The study of graph labelings diversified in 1963 when Sedlacek [8], drawing inspiration from magic squares in number theory, introduced a separate category of labelings known as magic labelings. Further broad- ening the classification of graphs, Cahit [3] defined the concept of a cordial labeling, a less restrictive condition than its graceful and harmonious coun- terparts. He provided foundational proofs that this labeling exists for many common graphs. [6] presents a short and focused survey on prime cordial and divisor cordial labelings of graphs. Numerous researchers have ex- plored various types of graph labeling, and a regularly updated literature survey on this topic can be found in [5]. Recently, Umapathi and Senthil Amutha [10] have examined a type of labeling known as Lipschitz labeling. They demonstrated that both star and comb graphs admit a Lipschitz labeling, while cycle and cyclic graphs do not permit a Lipschitz labeling. In this work, we investigate the existence and properties of Lipschitz labelings

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applied to certain classes of star graphs. A Lipschitz labelling is a function defined on the vertices (and/or edges) of a graph that satisfies a specific boundedness condition, ensuring that the difference between labels of adjacent vertices remains constrained. Our study aims to charac- terize the conditions under which such labelings exist for star graphs and to explore their structural implications. To systematically address these objectives, the paper is structured as follows: Section 2 establishes the foundational concepts, including key definitions, notation, and prior results related to graph labelings, with a focus on Lipschitz-type constraints. We review essential graph-theoretic notions and introduce the formal framework for analyzing star graphs in this context. We present our main contributions, examining the existence and construction of Lipschitz labelings for specific families of star graphs in Section 3. The final section summarizes the implications of our results, dis- cusses potential extensions to related graph classes, and outlines directions for future research.

2. BASIC DEFINITIONS

This section provides the essential definitions and concepts required to establish the framework for our study. We begin by formalizing key graph- theoretic notions, especially focusing on labeling schemes. The definitions presented here serve as the groundwork for understanding and analyzing the main results discussed in the subsequent sections. Building upon the classical notion of irregular labelings, Umapathi and Senthil Amutha [10] introduced the concept of a Lipschitz labeling for arbitrary graphs. Formally, the labelling is defined as follows:

Definition 1 *[1] A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set D in* R *if there exists a constant $L > 0$ such that*

$$|f(t, y_1) - f(t, y_2| \leq L|y_1 - y_2|$$

whenever both points (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f

Definition 2 *[10] Let $G = (V(G), E(G))$ be a finite simple graph with $|V(G)| = m$ and $|E(G)| = n \geq 1$. The function f: $V(G) \cup E(G) \to \{1, 2, \ldots, m + n\}$ is said to be Lipschitz labeling if f satisfies the following conditions.*

(i) *f is bijective.*

(ii) *$f(v) \in \{1, 2, \ldots, m\}$ for all $v \in V(G)$*

(iii) *$f(x) \in \{m + 1, m + 2, \ldots, m + n\}$ for all $x \in E(G)$*

(iv) *There exists a positive integer L such that*

$$|f(x) - f(y)| \leq L|x_f - y_f|$$

for all $x, y \in E(G)$ and $x_f = \min\{f(u), f(v)\}$ for $x = uv$.

Remark 1 *In the above defnition, we assume that x and y are need not be distinct.*

Definition 3 *A fork graph with two prongs and height $h \geq 1$ is denoted by $F_{2,h}$ and is obtained by joining $K_{1,2}$ with P_h.*

Example 1 *The fork graph with two prongs and height* 5 *is given by*

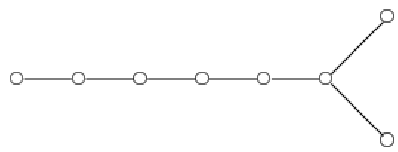

Figure 1: $F_{2,5}$

DOI: 10.5281/zenodo.17556520
Vol: 62 | Issue: 11 | 2025

Definition 4 *A h-graph with ascender $a \geq 1$ is denoted by h_a and is obtained by joining P_4 with P_a.*

Example 2 *The h-graph with ascender 3, that is h_3 is given by*

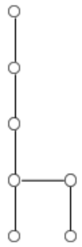

Figure 2: h_3

Definition 5 *A tree T is called m-ary tree if and only if T has eaxtly one vertex of degree m, called the root of T and other vertex, the degree is 1 or $m + 1$.*

Definition 6 *[9] A tree T is called full binary tree if and only if T has eaxtly one vertex of degree m, called the root of T and other vertex, the degree is 1 or 3. A full binary tree is called perfect binary tree if $2^{l+1} - 1$ vertices and $l \geq 0$ is called the level of the tree.*

Example 3 *[9] A perfect binary tree with level 3 is given by*

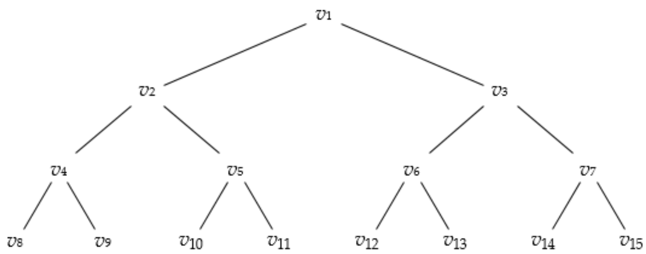

Figure 3: Perfect binary tree with level 3

Definition 7 *A flag graph of length $m \geq 1$ is a graph obtained from P_3 by adjoing two P_m with top two vertices of P_3.*

Example 4 *A flag graph with length 4 is given by*

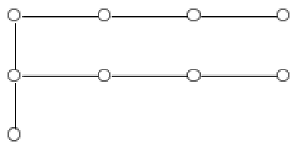

Figure 4: Flag graph with length 4

3. MAIN RESULTS

In this ection, we present our core contributions, examining the existence and construction of Lipschitz labelings for specific families of star graphs in Section 3. Theoretical proofs, illustrative examples, and comparative analyses are provided to validate the findings.

Theorem 1 *The fork graph $F_{2,n}$ ($n \geq 1$) is a Lipschitz graph.*

Proof

Clearly $K_{1,2} = P_3$ are Lipschitz graph with labeling discussed in [10]. Assume that $n > 2$.

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Vol: 62 | Issue: 11 | 2025

Define

$f\colon V(G) \cup E(G) \to \{1, 2, \ldots, 2n+5\}$,

as given below.

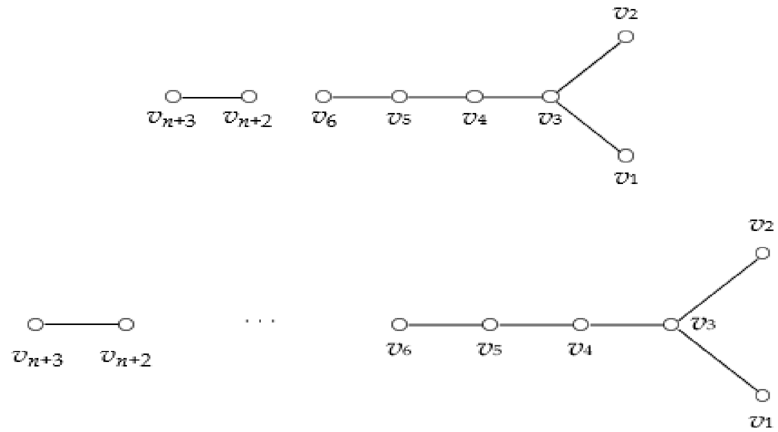

Figure 5: $F_{2,n}$

Let

$f(v_i) = i$,

and

$f(v_iv_j) = 2n + 6 - \min\{i, j\}$,

Let $x = v_iv_j$ and $y = v_kv_l$ be any two egdes in the graph.

Then

$$\begin{aligned} |f(x) - f(y)| &= |2n + 6 - \min\{i, j\} - (2n + 6 - \min\{k, l\})|, \\ &= |\min\{k, l\} - \min\{i, j\}| \\ &= \min\{f(v_k), f(v_l)\} - \min\{f(v_i), f(v_j)\}, \qquad \text{since } f(v_n) = n \\ &= y_f - x_f, \\ &= x_f - y_f, \end{aligned}$$

From the above discussion, we have

$|f(x) - f(y)| \le L|x_f - y_f|$,

for all $x, y \in E(G)$ with $L = 1$.

Hence $F_{2,n}$ is a Lipschitz graph.

Example 5 *A Lipschitz labeling for* $F_{2,5}$ *is given by*

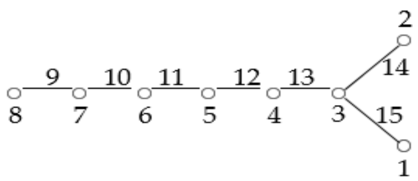

Figure 6: $F_{2,5}$ with its a Lipschitz laebling

DOI: 10.5281/zenodo.17556520
Vol: 62 | Issue: 11 | 2025

Theorem 2 *A h-graph with ascender a, h_a ($a \geq 1$) is a Lipschitz graph.*

Proof

Define

$$f : V(G) \cup E(G) \to \{1, 2, \ldots, n+4\}$$

as given below.

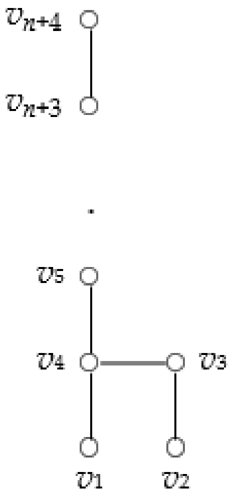

Figure 7: A *h*-graph with ascender *a*

Let

$$f(v_i) = i,$$

and

$$f(v_i v_j) = n + 4 + \min\{i, j\},$$

Let $x = v_i v_j$ and $y = v_k v_l$ be any two egdes in the graph.

Then

$$\begin{aligned} |f(x) - f(y)| &= |\, n + 4 + \min\{i, j\} - (n + 4 + \min\{k, l\})\,|, \\ &= |\min\{k, l\} - \min\{i, j\}\,|, \\ &= \min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_l)\}\ , \text{since } f(v_n) = n \\ &= x_f - y_f \end{aligned}$$

From the above discussion, we have

$$|f(x) - f(y)| \leq L \cdot |x_f - y_f|$$

for all $x, y \in E(G)$ with $L = 1$.

Hence h_a is a Lipschitz graph.

Example 6 *A Lipschitz labeling of* h_3 *is given by*

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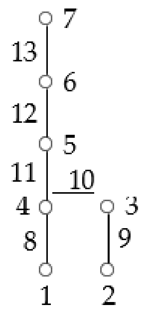

Figure 8: h_3 with its a Lipschitz laebling

Theorem 3 *The perfect binary tree T^l_B ($l \geq 0$) is a Lipschitz graph.*

Proof

Let G be a perfect binary tree T^l_B.

Let

$v_{2^{l+1}-1}$ – the root vertex,

$v_{2^{l+1}-3}$ – left child in the level 1,

$v_{2^{l+1}-2}$ – right child in the level 1,

v_1 – left child of the parent vertex v_{2^l+1} in the level l,

v_2 – right child of the parent vertex v_{2^l+1} in the level l,

v_{2^l-1} – left child of the parent vertex $v_{3(2^l+1)}$ in the level l,

v_{2^l} – right child of the parent vertex $v_{3(2^l+1)}$ in the level l.

Define

$$f\colon V(G) \cup E(G) \to \{1, 2, \ldots, 2^{l+2} - 3\}$$

as given below.

$$f(v_i) = i,\ i = 1, 2, \ldots, 2^{l+1} - 1,$$

$$f(v_i v_j) = 2^{l+1} - 1 + \min\{f(v_i), f(v_j)\}.$$

Let $x = v_i v_j$ and $y = v_k v_m$ be any two egdes in the graph.

Then

$$\begin{aligned} |f(x) - f(y)| &= |\ 2^{l+1} - 1 + \min\{f(v_i), f(v_j)\} - 2^{l+1} - 1 + \min\{f(v_k), f(v_m)\}\ |, \\ &= \min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_m)\}\text{ , since } f(v_n) = n \\ &= x_f - y_f\ . \end{aligned}$$

From the above two discussion, we have

$$|f(x) - f(y)| \leq L|x_f - y_f|$$

for all $x, y \in E(G)$ with $L = 1$.

Hence T^l_B is a Lipschitz graph.

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Vol: 62 | Issue: 11 | 2025

Example 7 *A Lipschitz labeling of T^3 is given by*

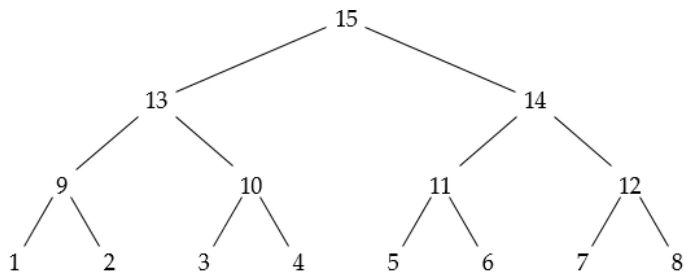

Figure 9: T_B^3 with its a Lipschitz laebling

Here, for instance,

the edge labbeling of the edge joining the vertices with lebel 13 and 10 $= 2^{3+1} - 1 + \min\{13, 10\}$,

$= 16 - 1 + 10 = 25$

Theorem 4 *A flag graph with length $m \geq 1$ is a Lipschitz graph.*

Let G be a flag graph with length $m \geq 1$.

Then G has $2m+3$ vertices and $2m+2$ edges.

Consider G as given below

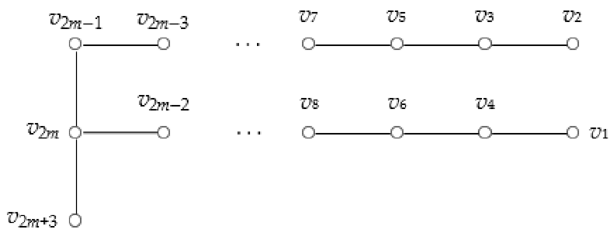

Figure 10: flag graph with length *m*

Define $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 2m+3, \ldots, 4m+5\}$ as

$$f(v_i) = i,$$

$$f(v_i v_j) = 2m + 3 + \min\{f(v_i), f(v_j)\},$$

for all $1 \leq i \leq 2m+3$.

Let $x = v_i v_j$ and $y = v_k v_m$ be any two egdes in the graph.

Then

$$|f(x) - f(y)| = |\ 2m + 3 + \min\{f(v_i), f(v_j)\} - \dot{}2m + 3 + \min\{f(v_k), f(v_m)\}\ |,$$

$$= \min\{f(v_i), f(v_j)\} - \min\{f(v_k), f(v_m)\}\text{ , since } f(v_n) = n$$

$$= x_f - y_f.$$

From the above discussion, we have

$$|f(x) - f(y)| \leq L|x_f - y_f|$$

DOI: 10.5281/zenodo.17556520
Vol: 62 | Issue: 11 | 2025

for all $x, y \in E(G)$ with $L = 1$.

Hence, flag graph is a Lipschitz graph.

Example 8 *A Lipschitz labeling of flag graph with length 4 is given by*

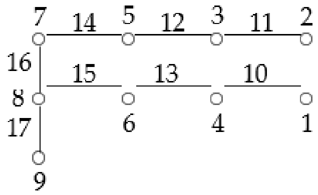

Figure 11: Flag graph with length 4 and it's a Lipschitz laebling

4. CONCLUSIONS

In this study, we investigate an algorithm for Lipschitz labelling applied to various graph structures, including a two-pronged fork graph, an h- graph, a perfect binary tree, and a flag graph. Through rigorous analysis, we demonstrate the algorithm's effectiveness in assigning labels while ad- hering to Lipschitz constraints, ensuring that adjacent vertices maintain bounded differences in their assigned values. Additionally, we provide an illustrative example to enhance conceptual understanding and validate the algorithm's applicability across discussed graphs. Our findings contribute to the broader discourse on Lipschitz labelling by establishing its feasibil- ity on diverse graph classes. We hope this work serves as a foundation for future research, encouraging further exploration of Lipschitz labeling techniques on more complex or specialized graph families.

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