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ANALYTICAL AND NUMERICAL SOLUTION USING VARIOUS ONE AND TWO STEPS METHOD OF GAUSS LOBATTO METHOD TO SOLVE FREDHOLM EQUATION

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Abstract

In this paper, we discuss numerical experiment of absolute stability of various one and two steps method for ODEs and Fredholm equation. The Gauss Lobatto method for approximating the value of the integrals presented. Two and one dimensional case of spectral element method are presented. The proposed method provides the solution in terms of convergent series with easily computable components, as well as it possesses main advantage as compared to other existed methods.

Keywords: *Numerical Method, Fredholm Equation, Gauss Lobatto Method.*

1. INTRODUCTION

A Fredholm Equation is defined as a type of nonlinear integral equation that involves functions within a specified domain and is commonly used in modern numerical methods for solving mathematical problems. In mathematics, the Fredholm integral equation is an integral equation whose solution gives rise to Fredholm theory, the study of Fredholm kernels and Fredholm operators. The integral equation was studied by Ivar Fredholm. A useful method to solve such equations, the Adomian decomposition method, is due to George Adomian. An inhomogeneous Fredholm equation of the second kind is given as

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt\ ,\ x \in [a,b]. \tag{1}$$

Given the kernel $K(x;t)$ and the function $f(x)$ the problem is typically to find the function $\varphi(x)$ The solution to a general Fredholm integral equation of the second kind is called an integral equation Neumann series. Fredholm equations arise naturally in the theory of signal processing, for example as the famous spectral concentration problem popularized by David Slepian.

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The operators involved are the same as linear filters. They also commonly arise in linear forward modelling and inverse problems. In physics, the solution of such integral equations allows for experimental spectra to be related to various underlying distributions, for instance the mass distribution of polymers in a polymeric melt, or the distribution of relaxation times in the system [5, 12, 13, 14,15, 16]. In addition, Fredholm integral equations also arise in fluid mechanics problems involving hydrodynamic interactions near finite-sized elastic interfaces. A specific application of Fredholm equation is the generation of photo-realistic images in computer graphics, in which the Fredholm equation is used to model light transport from the virtual light sources to the image plane. The Fredholm equation is often called the rendering equation in this context. Lobatto methods for the numerical integration of differential equations are named after Rehuel Lobatto. They are characterized by the use of approximations to the solution at the two end points t_n and t_{n+1} of each subinterval of integration $[t_n, t_{n+1}]$: In numerical analysis, an $n-$point Gaussian quadrature rule, named after Carl Friedrich Gauss is a quadrature rule constructed to yield an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the nodes x_i and weights ω_i for i = 1, …, n: The spectral element method (SEM) is a formulation of the finite element method (FEM) that uses high-degree piecewise polynomials as basis functions. The spectral element method was introduced in a 1984 paper by A. T. Patera.

In mathematics, in the area of numerical analysis, Galerkin methods are a family of methods for converting a continuous operator problem, such as a differential equation, commonly in a weak formulation, to a discrete problem by applying linear constraints determined by finite sets of basis functions. They are named after the Soviet mathematician Boris Galerkin [5, 12, 13, 14, 15, 16].

This paper is organized as follows: Section 2 introduces the Gauss Lobatto method for approximating. Section 3 presented the viscosity spectral element method - 1D. Section 4 deals the spectral element method - 2D and 1-element and weak formulation. Section 5 we presented the spectral Element Method of 2D and 1 element.

2. GAUSS LOBATTO METHOD FOR APPROXIMATING

Gauss Lobatto is a method for approximating the value of the integrals $\int_a^b f(x)dx,$ where f is a given; function through a linear transformation, the above integral can be reduced to $\int_{-1}^1 f(x)dx,$ where f not the same function. In this case Lobatto rule of the function f on [—1,1] is

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i)$$

where x_i is the $(i-1)^{\text{th}}$ root derivative Legendre polynomial $\frac{d}{dx}L_{n+1}(x).$ Recall that the recursive relation to define the Legendre polynomial is:

$$\begin{aligned} (n+1)L_{n+1}(x) &= (2n+1)xL_n(x) - nL_{n-1}(x) \\ L_0(x) &= 1 \\ L_1(x) &= x \end{aligned}$$

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and the weight ω_i is given by

$$w_i = \begin{cases} \frac{2}{n(n-1)[p_{n-1}(x_i)]^2} & ; \quad x_i \neq \pm 1 \quad , i \neq 1, n \\ \frac{2}{n(n+1)} & if \quad x_i = \pm 1 \quad (i.e \quad i = 1 \; or, \quad i = n). \end{cases} \tag{2}$$

In this method (Gauss Lobatto), the points x_i are chosen such that

$$-1 = x_1 < x_2 < \cdots < x_n = 1.$$

The boundary value problem that will be considered in in our study is:

$$\begin{cases} -u''(x) + u(x) & = f(x) \\ u(-1) & = 0 \\ u(1) & = 0. \end{cases} \tag{3}$$

To get the coefficients $u_i, \quad i = 0, \ldots, N$ we apply three different methods: spectral collocation method, spectral method of Galerkin and integral method in all cases we get a system that must be solved *AU = F.* The solution of (3) is expanded using Lagrange interpolations based on the Gauss-Lobatto Legendre points $u_N(x) = \sum_{i=0}^{N} u_i h_i(x)$, where

$$h_i(x) = \frac{(1-x^2)L'_N(x)}{N(N+1)L_N(x_i)(x-x_i)} \tag{4}$$

where $x_0 = -1$, $x_N = 1$ and u_i, $i = 0, ..., N$ are the unknowns coefficients and they are approximation of $u(x_i), i = 0, ..., N$ *i.e.* $u_i \simeq u(x_i)$. $u_N(x) = \sum_{i=0}^{N} u_i h_i(x)$, we will determine u_N such that the residual $R_N(x) = -u_N(x)'' + u_N(x) - f(x)$. Our solution is defined as becomes equals to zero at the interior points x_i, $i = 1, ..., N-1$ and u_N satisfies exactly the boundary conditions, i.e $u_N(-1) = 0 = u(-1)$ and $u_N(1) = 0 = u(1)$ where

$$h_i(x_k) = \delta_{ki} = \begin{cases} 1 & if \quad i = k \\ 0 & if \quad i \neq k. \end{cases}$$

The problem

$$\begin{cases} -u(x)'' + u(x) = f(x) \Rightarrow -\sum_{i=0}^{N} u_i h_i''(x) + \sum_{i=0}^{N} u_i h_i(x) = f(x) \\ u(-1) = 0 \Rightarrow \sum_{i=0}^{N} u_i h_i(-1) = 0 \\ u(1) = 0 \Rightarrow \sum_{i=0}^{N} u_i h_i(1) = 0. \end{cases} \tag{5}$$

We would like that $-u''_N(x) + u_N(x) = f(x)$ residual at interior points i.e x_j, $j = 1, ..., N-1$

$$-\sum_{i=0}^{N} u_i h_i''(x_j) + \sum_{i=0}^{N} u_i h_i(x_j) = f(x_j), \quad j = 1, ..., N-1 \tag{6}$$

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we have *N + 1* unknowns and *N - 1* equations. Unfortunately, the previous system cannot be solved since the number of equations is large then the number of unknowns, to solve this problem we add two equations comes from boundary conditions [5, 12, 13, 14, 15, 16].

2.1 Boundary Conditions

First boundary conditions is

$$\sum_{i=0}^{N} u_i h_i(-1) = 0 \Leftrightarrow \sum_{i=0}^{N} u_i h_i(x_0) = 0$$

$$\Rightarrow u_0 h_0(x_0) = 0$$

using $h_0(x_0) = 1 \Rightarrow u_0 = 0.$

Second boundary conditions:

$$\sum_{i=0}^{N} u_i h_i(1) = 0 \Rightarrow \sum_{i=0}^{N} u_i h_i(x_N) = 0$$

using

$$h_i(x_N) = \begin{cases} 1 & if \quad i = N \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} \sum_{i=0}^{N} u_i h_i(x_N) = 0 \\ u_N h_N(x_N) = & \quad u_N \\ u_N = & \quad 0. \end{aligned}$$

Now, we have

$$\begin{cases} -\sum_{i=0}^{N} u_i h_i''(x_j) + \sum_{i=0}^{N} u_i h_i(x_j) & = f(x_j) \quad j = 1, ..., N-1 \\ u_0 & = 0 \\ u_N & = 0. \end{cases}$$

$N+1$ equations and $N+1$ unknowns the system can be solved for $j = 1, ..., N-1$

$$-\sum_{i=0}^{N} u_i h_i''(x_j) + \sum_{i=0}^{N} u_i h_i(x_j) = f(x_j)$$

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$$-\left[u_0h_0''(x_j)+\sum_{i=1}^{N-1}u_ih_i''(x_j)+u_Nh_N''(x_j)\right]+\sum_{i=0}^{N}u_ih_i(x_j)+u_0h_0(x_0)+u_Nh_N(x_0)=f(x_j)$$

$$-\sum_{i=1}^{N-1}u_ih_i''(x_j)+\sum_{i=0}^{N-1}u_ih_i(x_j)=f(x_j);\quad j=1,...,N-1. \tag{7}$$

Let $h_{ji}=h_i''(x_j)\ \ i,j=1,\ldots,N-1$

$$D=(h_{ji}).\ \ j,i=1,...,N-1 \tag{8}$$

then

$$\left(\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N-1}\\ h_{21} & h_{22} & \cdots & h_{2N-1}\\ \vdots & \vdots & \ddots & \vdots\\ h_{n1} & h_{n2} & \cdots & h_{N-1N-1}\end{bmatrix}+Id_{N-1}\right)\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_{N-1}\end{bmatrix}=\begin{bmatrix} f(x_1)\\ f(x_2)\\ \vdots\\ f(x_{N-1})\end{bmatrix}.$$

In matrix form:

$$\begin{cases} (-D+Id)U & =F\\ U & =(u)^T_{i=1,\ldots,N-1}\\ F & =(f(x_i))^T_{i=1,\ldots,N-1}.\end{cases}$$

Now, compute the coefficients of D are:

$$h_i''(x)=\frac{1}{N(N+1)L_N(x_i)}\left[\frac{(1-x^2)}{x-x_i}L_N'(x)\right]''.$$

Now, we present a simple form of the coefficients of D

$$\begin{aligned}\left[\frac{(1-x^2)L_N'(x)}{x-x_i}\right]' &= \frac{\left[(1-x^2)L_N'\right]'(x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2}\\ &= \frac{\left[-2xL_N'(x)+(1-x^2)L_N''(x)\right](x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2}\end{aligned}$$

we recall that

$$(1-x^2)L_N''(x)-2xL_N'=-N(N+1)L_N(x)$$

then

$$\left(\frac{(1-x^2)L_N'(x)}{x-x_i}\right)'=\frac{-N(N+1)L_N(x)(x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2}$$

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then

$$\begin{aligned} h_i'(x) &= \frac{1}{N(N+1)L_N(x_i)} \frac{-N(N+1)L_N(x)(x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2} \\ h_i'(x) &= \frac{1}{N(N+1)L_N(x_i)} \left(\frac{L_N(x)}{x-x_i} + \frac{(1-x^2)L_N'(x)}{x-x_i} \right) \end{aligned}$$

$$\begin{aligned} h_i''(x) &= -\frac{1}{N(N+1)L_N(x_i)} \left(\left(\frac{L_N(x)}{x-x_i} \right)' + \left(\frac{(1-x^2)L_N'(x)}{x-x_i} \right)' \right) \\ \left(\frac{L_N(x)}{x-x_i} \right)' &= \frac{L_N'(x)(x-x_i)-L_N(x)}{(x-x_i)^2} \\ \left(\frac{(1-x^2)L_N'(x)}{x-x_i} \right)' &= -\frac{N(N+1)L_N(x)(x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2} \end{aligned}$$

$$h_i''(x) = -\frac{1}{N(N+1)L_N(x_i)} \left(\frac{L_N'(x)(x-x_i)-L_N(x)-N(N+1)L_N(x)(x-x_i)-(1-x^2)L_N'(x)}{(x-x_i)^2} \right)$$

$$h_i''(x) = -\frac{1}{N(N+1)L_N(x_i)} \frac{[(x-x_i)-(1-x^2)]L_N'(x)-[1+N(N+1)(x-x_i)]L_N'(x)}{(x-x_i)^2}. \qquad (9)$$

Now, the matrix equation will be solved to obtain the approximation solution. Note that if the size of the matrix is large enough one can consider iterative methods. In our numerical presentation, we will present both cases, this allow the reader to make a difference between small *N* and large *N*. Moreover, we will discuss later some numerical schemes to get a rapid solution (in iteration counts).

3. THE VISCOSITY SPECTRAL ELEMENT METHOD - 1D

Problem:

$$-u''(x) = f(x), \quad u(-1) = u(+1) = 0 \qquad (10)$$

We will use the spectral method to find the approximate solution using " Gauss Lobbato points" we define the set

$$V^N = \{v \in P_N([-1,1]) : v(-1) = v(1) = 0\}$$

where $P_N([-1; 1])$ is the space of polynomials of degree at most *N* on [-1,1]. We multiply both sides of equation (10) with $v \in V^N$ as test function, we get

$$\int_{-1}^{1} -u''(x)v(x)dx = \int_{-1}^{1} f(x)v(x)dx$$

3.1 Spectral element method 1D: Weak formulation

$$\int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} f(x)v(x)dx.$$

The approximate solution is expanded using lagrange interpolants based on the Gauss Lobatto Legendre points [5, 12, 13, 14, 15, 16]:

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$$u_N(x) = \sum_{i=0}^{N} c_i h_i(x)$$

where

$$h_i(x) = \frac{(1-x^2)L'_N(x)}{N(N+1)L_N(x_i)(x-x_i)}.$$

Spectral element method - 1D

- $x_i = 0, 1, \ldots, N$ are Gauss Lobatto Legandre Points with $x_0 = -1$ and $x_N = 1$
- c_i: are the unknown coefficients with $u(x_i) \simeq u_N(x_i) = c_i$

• Gauss-Lobatto Quadrature rule:

$$\int_{-1}^{1} \phi(x)dx = \sum_{i=0}^{N} w_i \phi(x_i)$$

x_i are roots of $(1-x^2)L'_N(x) = 0$. The weights w_i given by

$$w_i = \frac{2}{N(N+1)L_N^2(x_i)} \quad i = 0, 1, \ldots, N.$$

3.2 Spectral element method - 1D

plugging u_N into the weak formulation and we choose $v(x) = h_j(x)$ we have:

$$\sum_{i=1}^{N-1} c_i \int_{-1}^{1} h'_i(x)h'_j(x)dx = \int_{-1}^{1} f(x)h_j(x)dx \quad i = 0, \ldots, N$$

using quadrature rule to compute both integrals we obtain:

$$\int_{-1}^{1} h'_i(x)h'_j(x)dx = \sum_{k=0}^{N} w_k h'_i(x_k)h'_j(x_k)$$

$$\int_{-1}^{1} f(x)h_j(x)dx = \sum_{k=0}^{N} w_k f(x_k)h_j(x_k),$$

then we have :

$$\sum_{i=1}^{N-1} c_i \left(\sum_{k=0}^{N} w_k h'_i(x_k)h'_j(x_k)\right) = \sum_{k=0}^{N} w_k f(x_k)h_j(x_k) \quad j = 1, \ldots, N-1$$

we have $N-1$ equations and $N-1$ unknowns:

$$AU = F, \quad A = (a_{ij})_{i,j=1,\ldots,N-1}, \quad F = (f_j)^T_{j=1,\ldots,N-1}$$

with

$$a_{ij} = \sum_{k=0}^{N} w_k h'_i(x_k)h'_j(x_k) \tag{11}$$

$$f_j = \sum_{k=0}^{N} w_k f(x_k)h_j(x_k) = w_j f(x_j) \tag{12}$$

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4. SPECTRAL ELEMENT METHOD - 2D AND 1-ELEMENT AND WEAK FORMULATION

Let the Poission equation

$$\begin{cases} -\Delta u &= f(x,y), \quad \text{in } \Omega = [1-,1]^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

We multiply by a test function and integrate over Ω: Find $u \in H_0^1(\Omega)$

$$\int_\Omega \nabla u(x,y)\nabla v(x,y)dxdy = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega).$$

Spectral element method - 2D satisfy the following:

1. Galerkin Approximation Let $V^N \subseteq H_0^1(\Omega)$ be a finite subspace of $H_0^1(\Omega)$. Find $U_N \in V^N$; $\forall v_N \in V^N, \quad \int_\Omega \nabla u_N \nabla v_N = \int_\Omega f v_N$

2. Discretization: Approximation of previous integrals using numeral integration rules

$$\int_\Omega \nabla u(x,y)\nabla v(x,y)dxdy = B_N(u_N, v_N)$$

$$\int_\Omega fv, \quad \forall v = L(v_N).$$

Then, we solve the system obtained

$$B_N(u_N, v_N) = L(v_N).$$

The spectral element method - 2D, (one element), using

$$u(x,y) = \sum_{i,j=1}^{N+1} u_{ij} h_i(x) h_j(y)$$

and we take as test function $v_N = h_k(x)h_l(y)$. Let $D_{kj} = h_j'(x_k), \quad f_{kl} = f(x_k, y_l)$

$$\sum_{i,j=1}^{N+1} u_{ij}\Big(\sum_{m=1}^{N+1} D_{mi}D_{km}\delta_{jl}w_m w_l \delta_{ml} + \sum_{n=1}^{N+1} D_{jn}D_{ln}\delta_{ik}w_k w_n \delta_{kn}\Big) = w_k w_l f_{kl}$$

5. SPECTRAL ELEMENT METHOD OF 2D AND 1 ELEMENT

The differentiation matrix defined as:

$$D_k j = \begin{cases} \frac{1}{(x_k - x_j}\frac{L_N(x_k)}{L_N(x_j)} & j \neq k \\ 0 & 1 \leq J = k \leq N-1 \\ \frac{N(N+1)}{4} & J = K = 0 \\ \frac{N(N+1)}{4} & J = K = N \end{cases}$$

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5.1 Spectral element method - 2D and 2- element

Let

$$\begin{aligned}(\xi,\eta)\in\Omega &= [-1,1]^2\\ \Lambda_\lambda:[-1,1]^2\to\Omega_\lambda &= [a_\lambda,b_\lambda]\times[c_\lambda,d_\lambda]\\ (\xi,\eta)\to(\alpha_\lambda\xi+\alpha_\lambda^{'};\beta_\lambda\eta+\beta_\lambda^{'})\end{aligned}$$

therefore

$$\alpha_\lambda=\frac{b_\lambda-a_\lambda}{2}\ ;\ \alpha_\lambda^{'}=\frac{a_\lambda+b_\lambda}{2}$$

and

$$\beta_\lambda=\frac{d_\lambda-c_\lambda}{2}\ ;\ \beta_\lambda^{'}=\frac{d_\lambda+c_\lambda}{2}$$

then the general form of the mapping is:

$$(\xi,\eta)\to\left(\frac{b_\lambda-a_\lambda}{2}\xi+\frac{a+b}{2},\frac{d_\lambda-c_\lambda}{2}\eta+\frac{d+c}{2}\right).$$

Let $x_\lambda=\alpha_\lambda\xi+\alpha^{'},\quad y_\lambda=\beta_\lambda\eta+\beta_\lambda^{'}$

$$\Lambda_\lambda(\xi,\eta)=(x_\lambda.y_\lambda)$$

$$\begin{aligned}J &= \begin{bmatrix}\frac{\partial x_\lambda}{\partial\xi} & \frac{\partial x_\lambda}{\partial\eta}\\ \frac{\partial y_\lambda}{\partial\xi} & \frac{\partial y_\lambda}{\partial\eta}\end{bmatrix}\\ &= \begin{bmatrix}\alpha_\lambda & 0\\ 0 & \beta_\lambda\end{bmatrix}\end{aligned}$$

two element Jacobian determinant $|J|=\alpha_\lambda\beta_\lambda$
weak formulation:

$$\int_\Omega\nabla u(x,y)\nabla v(x,y)=\int_\Omega f(x,y)v(x,y)\partial x\partial y\quad\forall v\in H_0^1(\Omega)$$

bilinear form

$$B(u,v)=\int_\Omega\nabla u\nabla v$$

linear form

$$L(v)\int_\Omega fv$$

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since we have two elements , we get:

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} fv \Rightarrow \sum_{\lambda=1}^{k} \int_{\Omega_\lambda} \nabla u \nabla v = \sum_{\lambda=1}^{k} \int_{\Omega_\lambda} fv$$

using affine mappings , the integral can be evaluated as for $\lambda = 1, 2, .., k$ (number of subdomains)

$$\int_{\Omega_\lambda} \nabla u \nabla v = \int_{\Omega} u_\lambda^\wedge(\xi, \eta) v_\lambda^\wedge(\xi, \eta) |J| \partial\xi \partial\eta$$

$$= \alpha_\lambda \beta_\lambda \sum_{i,j=0}^{N} u_\lambda^\wedge(\xi, \eta) v_\lambda^\wedge(\xi, \eta) w_i w_j.$$

Two element: weak formulation:

$$\int_{\Omega} \nabla u(x, y) \nabla v(x, y) = \int_{\Omega} f(x, y) v(x, y) \partial x \partial y \ , \ \ \forall v \in H_0^1(\Omega)$$

bilinear form

$$B(u, v) = \int_{\Omega} \nabla u \nabla v$$

linear form $L(v) = \int_\Omega fv$ since we have two elements, we get:

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} fv \Rightarrow \sum_{\lambda=1}^{k} \int_{\Omega_\lambda} \nabla u \nabla v = \sum_{\lambda=1}^{k} \int_{\Omega_\lambda} fv$$

using affine mappings , the integral can be evaluated as for $\lambda = 1, 2, ..., k$ (number of subdomains)

$$\begin{aligned}
\int_{\Omega_\lambda} \nabla u \nabla v &= \int_{\Omega} u_\lambda^\wedge(\xi, \eta) v_\lambda^\wedge(\xi, \eta) |J| \partial\xi \partial\eta \\
&= \alpha_\lambda \beta_\lambda \sum_{i,j=0}^{N} u_\lambda^\wedge(\xi, \eta) v_\lambda^\wedge(\xi, \eta) w_i w_j \\
\int_{\Omega_\lambda} f(x, y) v(x, y) \partial x \partial y &= \alpha_\lambda \beta_\lambda \sum_{i,j=0}^{N} f_\lambda^\wedge(\xi_i, \eta_j) h_l(\xi_i) h_s(\eta_j) w_i w_j
\end{aligned}$$

5.2 Two and Four elements of subdomains Ω_1 and Ω_2

Let

$$\begin{cases} -\Delta u = f \quad in\, \Omega = [-1, 1] \times [-1, 1] \\ \qquad u = 0 \quad on \quad \partial\Omega \end{cases}$$

where *(x; y)* $\in \Omega$. We decompose Ω into two non-overlapping subdomains Ω_1 and Ω_2

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$$\Omega_1 = [-1,0] \times [-1,1]$$

$$\Omega_2 = [0,1] \times [-1,1]$$

four elements

$$\Omega_1 = [-1,-\frac{1}{2}] \times [-1,1]$$

$$\Omega_2 = [-\frac{1}{2},0] \times [-1,1]$$

$$\Omega_3 = [0,\frac{1}{2}] \times [-1,1]$$

$$\Omega_4 = [\frac{1}{2},1] \times [-1,1].$$

Let *K* the number of subdomains. In our work *K = 2* (2 elements), *k = 4* (4 elements), we define the transformation

$$\Lambda_k : [-1,1] \times [-1,1] \to \Omega_k$$
$$(\xi,\eta) \mapsto (\alpha_k \xi + \alpha_k^{'}, \beta_k \eta + \beta_k^{'})$$
$$(x_i^k, y_j^k) = \Lambda_k(\xi_i,\eta_j), k = 1,\cdots,K$$

for example for 2 elements:

$$\Lambda_k : [-1,1] \times [-1,1] \to \Omega_1 = [-1,0] \times [-1,1]$$

$$(\xi,\eta) \mapsto (\alpha_k \xi + \alpha_k^{'}, \beta_k \eta + \beta_k^{'}).$$

its easy to see that $\beta_k = 1, \beta_k^{'} = 0$ for αk_1 and $\alpha_k^{'}$ we solve the following system:

$$\Lambda_1(-1,\eta) = (-1,\eta) \ ; \ \Lambda_1(1,\eta) = (0,\eta)$$

$$\begin{aligned} -\alpha + \alpha^{'} &= -1 \\ \alpha + \alpha^{'} &= 0. \end{aligned}$$

So, $\alpha = \frac{1}{2}, \alpha^{'} = -\frac{1}{2}$ then:

$$\Lambda_1 : [-1,1] \times [-1,1] \to \Omega_1$$

$$(\xi,\eta) \mapsto (\frac{1}{2}(\xi - 1), \eta)$$

we can done for Λ_2

$$\Lambda_2 : [-1,1] \times [-1,1] \to \Omega_2 = [0,1] \times [-1,1]$$
$$(\xi,\eta) \mapsto (\alpha_2 \xi + \alpha_2, \eta)$$

$$\begin{aligned} \Lambda_2(-1,\eta) &= (0,\eta) \\ \Lambda_2(1,\eta) &= (1,\eta) \end{aligned}$$

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we deduce

$$\begin{aligned} -\alpha_2 + \alpha_2^{'} &= 0 \\ \alpha_2 + \alpha_2^{'} &= 1. \end{aligned}$$

So, $\alpha_2 = \alpha_2' = \frac{1}{2}$ Then:

$$\Lambda_2 : [-1,1] \times [-1,1] \to \Omega_2$$

$$(\xi, \eta) \mapsto (\frac{1}{2}(\xi + 1), \eta).$$

We solve:

$$\begin{cases} -\Delta u = f \quad in\, \Omega \\ u = 0 \quad on\, \partial\Omega \end{cases}$$

$$\begin{cases} -\Delta u^{'} = f \quad in\, \Omega \\ u^1 = 0 \quad on\; \partial\Omega \cap \partial\Omega_1 \\ u^1 = u^2 \, on\, \partial\Omega \cap \partial\Omega_1 \\ \frac{\partial u^{'}}{\partial n} = \frac{\partial u^2}{\partial n} \end{cases}$$

and

$$\begin{cases} -\Delta u^2 = f \quad in\, \Omega_2 \\ u^2 = 0 \quad on\; \partial\Omega \cap \partial\Omega_2 \\ u^2 = u^1 \, \frac{\partial u^2}{\partial n} = \frac{\partial u^1}{\partial n} \\ on\;\; \partial\Omega_1 \cap \partial\Omega_2 \end{cases}$$

Let

$$\Gamma_{kk^1} = \partial\Omega_k \cap \partial\Omega_k^1,$$

$$\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2.$$

Space of approximation for $N \in \mathbb{N}$;

$$Y_N = \begin{cases} w \in L^2(\Omega); w\mid_{\Omega_k} \in \mathbb{P}_N(\Omega_k) \\ k = 1, \cdots, K \end{cases}$$

space of piecewise polynomial the Dirichlet problem:

$$\begin{cases} -\triangle u = f \;\, in \;\; \Omega \\ u = 0 \;\; on \;\; \partial\Omega \end{cases}$$

The solution of approximation

$$X_N = Y_N \cap H_0^1(\Omega)$$

$$X_N = \left\{ w \in Y_N, w = 0 \;\; on\, \partial\Omega, \forall kk^1, w^k\mid_{\Gamma_{kk^1}} = w_1^{k^1}\mid_{kk^1} \right\}$$

where $w^k \mid_{\Gamma_{kk^1}}$: denote the restriction of w^k on Γ_{kk^1} (interface)

$$X_N = \left\{ Y_{N^1}, if(x_i^k, y_j^k) \in \partial\Omega \;\; then \;\; w^k(x_i^k, y_j^k), if(x_i^k, y_j^k) = w^{k^1}(x_i^{k^1}, y_j^{k^1}) \right\}.$$

First: How can compute integral over Ω

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$$\int_{\Omega}\varphi(x,y)dxdy=\sum_{i=0}^{N}\sum_{j=0}^{N}\varphi(x_i^k,y_j^k)\alpha_k\beta_k w_i w_j$$

where $\alpha_k\beta_k$ Jacobian and w_iw_j is weight. We know that $(x,y)=\Lambda_k(\xi,\eta)$

$$J_k(\xi,\eta)=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta}$$

$$\begin{aligned} x &= \alpha_k\xi+\alpha k^1 \\ y &= \beta k\eta+\beta k^1 \end{aligned}$$

that implies

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \alpha k\ ,\ \frac{\partial x}{\partial \eta}=0 \\ \frac{\partial y}{\partial \eta} &= \beta k,\ \frac{\partial y}{\partial \eta}=1 \end{aligned}$$

then

$$J_k(\xi,\eta)=\begin{vmatrix}\alpha_k & 0\\ 0 & \beta_k\end{vmatrix}=\alpha_k\beta_k$$

for two elements: $J_k(\xi,\eta)=\frac{1}{2}$. Let us determine the transformation for 4-elements

$$\Lambda_1:[-1,1]\times[-1,1]\to[-1,-\frac{1}{2}]\times[-1,1]$$

$$(\xi,\eta)\mapsto(\alpha_1\xi+\alpha_1^{'},\eta)$$

$$\begin{aligned} \Lambda_1(-1,\eta) &= (-1,\eta) \\ \Lambda_1(1,\eta) &= (-\frac{1}{2},\eta) \end{aligned}$$

that implies

$$\begin{aligned} -\alpha_1+\alpha_1^{'} &= -1 \\ \alpha_1+\alpha_1^{'} &= -\frac{1}{2} \end{aligned}$$

so, $\alpha_1=\frac{1}{4}$ and $\alpha_1^{'}=-\frac{3}{4}$

$$\Lambda_1:[-1,1]\times[-1,1]\to\left[-1,-\frac{1}{2}\right]\times[-1,1]$$

$$(\xi,\eta)\mapsto\left(\frac{1}{4}(\xi-3),\eta\right)$$

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$$\Lambda_2 : [-1,1]\times[-1,1] \to [-\frac{1}{2}, \times[-1,1]$$
$$(\xi,\eta) \mapsto (\alpha_2\xi + \alpha_2^{'}, \eta)$$

$$\begin{aligned} \Lambda_2(-1,\eta) &= (-\frac{1}{2},\eta) \\ \Lambda_2(1,\eta) &= (0,\eta) \end{aligned}$$

then

$$\alpha_2^{'} = -\frac{1}{4}$$
$$\alpha_2 = \frac{1}{4}$$

then

$$\Lambda_2 : [-1,1]\times[-1,1] \hookrightarrow [-\frac{1}{2},0]\times[-1,1]$$
$$(\xi,\eta) \mapsto (\frac{1}{4}(\xi-1),\eta).$$

Similarly we, can get Λ_3 and Λ_4

$$\Lambda_3 : [-1,1]\times[-1,1] \hookrightarrow [0,\frac{1}{2}]\times[-1,1]$$
$$(\xi,\eta) \mapsto (\frac{1}{4}(\xi+1),\eta).$$

and

$$\Lambda_4 : [-1,1]\times[-1,1] \hookrightarrow [\frac{1}{2},1]\times[-1,1]$$
$$(\xi,\eta) \mapsto (\frac{1}{4}(\xi+3),\eta).$$

We define

$$\mathbb{P}_N^0(\Omega) = \{v \in \mathbb{P}_N(\Omega),\ \ v = 0 \ \ on \ \ \partial\Omega\}.$$

Finally, we get $\mathbb{P}_N$ is the space of polynomial of degree N, and $X_N = \mathbb{P}_N^0(\Omega) \subset H_0^1(\Omega)$.

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